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MONTHLY PROGRESS REPORT # 1

for

SPACE SHUTTLE PROPULSION PARAMETER ESTIMATION

using

OPTIMAL ESTIMATION TECHNIQUES



submitted to

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION  
MARSHALL SPACE FLIGHT CENTER  
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## 1.0 INTRODUCTION

This first monthly progress report details the mathematical developments necessary for the extended Kalman filtering and modified Bryson-Frazier smoothing algorithms. This is an expansion on the material presented at the initial meeting on 3 May 1983 including the current mathematical developments.

In this initial monthly progress report, only the first twelve system state variables are presented with the necessary mathematical developments for incorporating them into the filter/smoothen algorithm. Other state variables, i.e., aerodynamic coefficients can be easily incorporated into the estimation algorithm, representing uncertain parameters, but for initial checkout purposes are treated as known quantities.

An approach for incorporating the NASA propulsion predictive model results into the optimal estimation algorithm has been identified. This approach utilizes numerical derivatives and nominal predictions within the algorithm with global iterations of the algorithm. The iterative process is terminated when the quality of the estimates provided no longer significantly improves.

## 2.0 FILTERING AND SMOOTHING ALGORITHM

The Space Shuttle Parameter Estimation Program utilizes optimal estimation techniques to provide estimates of the propulsion system parameters. The technique selected is the extended Kalman filter and the modified Bryson-Frazier smoother. By modeling the propulsion system parameters as time correlated random variables, improved estimates of these parameters are obtained and are properly time phased by removing the filter induced lag by using the combined filter/smoothing. The smoother also provides improved estimates of the initial state estimates.

The system, in state-space notation, is modeled as the continuous dynamical system disturbed by additive Gaussian white noise

$$\dot{\underline{x}} = \underline{f}(\underline{x}(t), t) + G(t) \underline{w}(t) + \underline{u}(t), \underline{x}(0) = \underline{x}_0 \quad (1)$$

where

$\underline{x}$  = n-dimensional state vector

$\underline{x}_0$  = Gaussian initial condition vector with covariance  $P_0$

$\underline{w}(t)$  = p-dimensional white, zero-mean white Gaussian noise with covariance

$$E[\underline{w}(t) \underline{w}^T(\tau)] = Q(t) \delta(t - \tau)$$

$\underline{u}(t)$  = n-dimensional control vector.

The elements of the vector  $\underline{x}(t)$  represent vehicle position, velocities, attitudes, angular rates, aerodynamic and propulsion parameters, measurement biases, etc. Elements of  $\underline{u}(t)$  include known control inputs such as SSME power level commands.

The system described by equation (1) is observed at discrete times,  $t_k$ , with not all states being directly measured. Some measurements are non-linear functions of the elements of the state vector  $\underline{x}(t)$ . In general the measurement process is described as

$$\underline{z}_k = \underline{h}_k(\underline{x}(t_k)) + \underline{v}_k \quad (2)$$

where

$\underline{z}_k$  = m-dimensional observation vector

$\underline{h}_k$  = functional representation of the measurements in terms of the states

$\underline{v}_k$  = m-dimensional, zero-mean, white Gaussian noise sequence with covariance

$$E[\underline{v}_i \underline{v}_j^T] = R_i \delta_{i,j}$$

Examples of the elements of the observation vector  $\underline{z}_k$  include radar measurements of range, azimuth, and elevation from the radar site to the vehicle.

It is assumed that the system process noise vector  $\underline{w}(t)$  and the measurement noise vector  $\underline{v}_k$  are uncorrelated. Also, the system state initial condition vector  $\underline{x}_0$  is not correlated with either of these two noise vectors. Therefore

$$E[\underline{w}(t) \underline{v}_k^T] = 0, \quad E[\underline{w}(t) \underline{x}_0^T] = 0, \quad E[\underline{x}_0 \underline{v}_k^T] = 0$$

where the superscript T denotes transpose. For later reference, the following matrices are defined

$$F(\underline{x}(t), t) = \frac{\partial \underline{f}(\underline{x}(t), t)}{\partial \underline{x}(t)} \quad (3)$$

and

$$H(\underline{x}_k(-)) = \frac{\partial h(\underline{x}(t_k))}{\partial \underline{x}(t_k)} \quad (4)$$

These matrices are linearizations of the dynamics and measurement models respectively, evaluated about either a nominal or reference value of the state, or about the state estimate.

## 2.1 Extended Kalman Filter Algorithm

The extended Kalman filter algorithm is in essence a conventional linear Kalman filter algorithm applied to a mathematical model resulting from the linearization of the system model equation (1), and measurement process, equation (2), about a current state estimate. The filter yields optimal estimates if the linearization is accurate, i.e., the state estimate closely approximates the true state. The derivation of the algorithm can be found in reference [1].

The algorithm proceeds as follows. After initialization of the state estimate and covariance, the state estimate and covariance are propagated forward in time until a measurement update is available, by

$$\dot{\underline{x}} = \underline{f}(\underline{x}(t), t) \quad , \quad t_{k-1} < t \leq t_k \quad (5)$$

and

$$\dot{P}(t) = F(\underline{x}(t), t) P(t) + P(t) F(\underline{x}(t), t)^T + G(t) Q(t) G(t)^T \quad (6)$$

At the measurement time, the state estimate and covariance are updated by

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k(z_k - h_k(\hat{x}_k(-))) \quad (7)$$

and

$$P_k(+) = (I - K_k H_k(\hat{x}_k(-))) P_k(-) \quad (8)$$

where the  $(-)$  and  $(+)$  represent the appropriate values just before and just after the update. The updated values are used to reinitialize the time propagation equations (3) and (4) for integrating up to the next measurement time. The Kalman gain matrix is computed as

$$K_k = P_k(-) H_k(\hat{x}_k(-))^T (P_k(-) H_k(\hat{x}_k(-))^T + R_k)^{-1} \quad (9)$$

This algorithm is repeated until the last time point,  $t_N$ , is processed. For later use in the smoother algorithm, various combinations of the state estimates ( $\hat{x}$ ), measurements ( $z$ ), linearized dynamics matrix ( $F$ ) and measurement matrix ( $H$ ), measurement noise covariance ( $R$ ) and estimation error covariance matrix ( $P$ ) must be stored for each time instant to be processed by the smoother algorithm.

## 2.2 Modified Bryson-Frazier Smoother Algorithm

The operation of the smoother algorithm is similar to the filter algorithm except in reverse time. The derivation of this smoother algorithm is found in reference [2]. This fixed interval smoothing algorithm provides optimal estimates given all the measurements in comparison to the filtering algorithm providing optimal estimates given the previous



measurements processed. Therefore the smoother provides improved estimates in addition to removing the time lag induced by the filter algorithm.

The smoothing algorithm adjoint variables,  $\underline{\lambda}$  and  $\Lambda$  are "initialized" at the final time processed by the filter,  $T$ ,

$$\underline{\lambda}(T-) = -H_N^T (H_N P_N H_N^T + R_N)^{-1} (\underline{z}_N - H_N(\hat{x}_N(-))) \delta_{t_{N,T}} \quad (10)$$

and

$$\Lambda(T-) = H_N (H_N P_N H_N^T + R_N)^{-1} H_N \delta_{t_{N,T}} \quad (11)$$

If  $T$  is not an observation time,  $\underline{\lambda}$  and  $\Lambda$  are zero. The adjoint variables are propagated in reverse time to the next previous measurement time by

$$\dot{\underline{\lambda}} = -F(\hat{x}(t), t) \underline{\lambda}, \quad t_k \leq t < t_{k+1} \quad (12)$$

$$\dot{\Lambda} = -F(\hat{x}(t), t)^T \Lambda - \Lambda F(\hat{x}(t), t) \quad (13)$$

At the time of an available measurement,  $t_k$ , the adjoint variables are updated by

$$\begin{aligned} \underline{\lambda}(-) = \underline{\lambda}(+) - H_k^T (H_k P_k H_k^T + R_k)^{-1} ((\underline{z}_k - h_k(\underline{x}_k(-))) \\ + (H_k P_k H_k^T + R_k) K_k^T \underline{\lambda}(+)) \end{aligned} \quad (14)$$

and

$$\Lambda(-) = (I - K_k H_k^T)^T \Lambda(+) (I - K_k H_k) + H_k^T (H_k P_k H_k^T + R_k)^{-1} H_k \quad (15)$$

The smoother state estimate and error covariance are obtained using the filter estimate and covariance and the adjoint variables by

$$\underline{x}^*(t) = \underline{\hat{x}}(t) - P(t) \underline{\lambda}(t) \quad (16)$$

and

$$P^*(t) = P(t) - P(t) \Lambda(t) P(t). \quad (17)$$

Due to the potential number of time points to be processed, smoother estimates may only be computed at the discrete measurement times. For this approach the propagation equations (10) and (11) are replaced by

$$\underline{\lambda}_k(+) = \Phi_k^T \underline{\lambda}_{k+1}(-) \quad (18)$$

and

$$\Lambda_k(+) = \Phi_k^T \Lambda_{k+1}(-) \Phi_k \quad (19)$$

where  $\Phi_k^T$  is the state transition matrix formed with the linearized dynamics matrix  $F$  to propagate the adjoint variable from time  $t_{k+1}$  to time  $t_k$ . The algorithm continues in reverse time until the initial time is reached.

### 2.3 Iterations with the Filter/Smoother Algorithm

The performance of the filter/smoother algorithm is a direct result of the accuracy of the linearization. Repeated operations of the algorithms with adjustments in initial state estimates and covariance in each cycle can yield improved estimates. This technique is known as global iterated

filtering as defined in reference [3]. Each cycle of operating the algorithms would yield increasing improvements in the state estimates.

This feature of the algorithm operation is of special interest to the propulsion parameter estimation problem using the NASA predictive models. Initial, or nominal, values of the parameters of interest can be used to obtain the necessary partial derivatives indicated earlier. From operating the algorithm improved estimates of those parameters are obtained. Using these improved estimates, more accurate partial derivatives are obtained for use in the algorithms. This process is continued until there is in essence no change in the partial derivatives or quality of the state estimates. If the linearization is accurate, the measurement residual should be a white noise process with known covariance.

### 3.0 FILTER/SMOOTHER ALGORITHM SYSTEM AND MEASUREMENT MODEL

The usefulness of the filter/smoothing algorithm is to provide estimates of the system states from the observed motion and dynamics while the system is driven by known and unknown elements. These unknown elements are elements of the system state vector to be estimated. The evolution of motion resulting from these known and unknown elements is assumed to be suitably represented for this study by a six degree-of-freedom (6 DOF) rigid body equations of motion. These equations are presented and discussed in section 3.1.

To implement these equations into the filter/smoothing algorithm presented in section 2.0, a linearization of the system state and measurement models is required. These linearized equations are presented in section 3.2.

### 3.1 Equations of Motion and Measurement Equations

#### 3.1.1 Rigid Body Equations of Motion

The rate of change of vehicle velocity in body coordinates,  $\underline{v}^{(B)}$ , as a result of external forces acting on the vehicle is described by

$$\dot{\underline{v}}^{(B)} = \frac{\rho A v_m^2}{2m} \underline{c}_f + {}^B C^I \underline{g}^{(I)}(\underline{r}^{(I)}) - \underline{\omega} \times \underline{v}^{(B)} + \frac{\underline{f}_T^{(B)}}{m} + \frac{\underline{f}_P^{(B)}}{m} \quad (20)$$

where

$\rho$  = atmospheric density

$A$  = aerodynamic coefficient referenced area

$v_m$  = magnitude of vehicle velocity relative to the surrounding air mass

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$m$  = vehicle mass

$\underline{c}_f$  = aerodynamic force coefficient vector

$\underline{g}^{(I)} (\underline{r}^{(I)})$  = gravity vector in inertial coordinates

$\underline{\omega}$  = angular rotation of the body relative to the inertial frame

$\underline{f}_T^{(B)}$  = resultant thrust force vector in body coordinates

$\underline{f}_p^{(B)}$  = resultant plume force vector in body coordinates

The rate of change of vehicle position in inertial coordinates,  $\underline{r}^{(I)}$ , is then obtained by

$$\dot{\underline{r}}^{(I)} = I_C^B \underline{v}^{(B)} \quad (21)$$

where  $I_C^B$  is the transformation matrix from body coordinates to inertial coordinates. The elements of the  $I_C^B$  transformation matrix are obtained from the resulting Euler angles defined by

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

where  $\phi$ ,  $\theta$ , and  $\psi$  are roll, pitch and yaw attitudes respectively.

The roll, pitch and yaw rates of the body relative to inertial coordinates are  $p$ ,  $q$ , and  $r$  respectively. Finally, the rate of change of the body rates relative to inertial is given by

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = [I]^{-1} \left[ \frac{\rho A V_m^2}{2} \underline{\ell c}_m + \frac{\rho A V_m^2}{2} (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \underline{c}_f \right. \\ \left. - \underline{\omega} \times (\underline{T}\underline{\omega}) + \underline{T}_T^{(B)} + \underline{T}_P^{(B)} \right] \quad (23)$$

where

$I$  = vehicle moments of inertia matrix

$\underline{\ell c}_m$  = aerodynamic coefficient referenced length and moment  
coefficient vector

$\underline{r}_{cg}^{(B)}$  = vehicle center-of-gravity vector in body coordinates

$\underline{r}_A^{(B)}$  = aerodynamic coefficient reference position in body coordinates

$\underline{T}_T^{(B)}$  = resultant thrust torque vector in body coordinates

$\underline{T}_P^{(B)}$  = resultant plane torque vector in body coordinates

The equations of motion represent the first twelve elements of the system state vector. These equations are summarized in Table 3.1.1-1.

The moment of inertia matrix  $I$  in general is given by

$$I = \begin{bmatrix} I_x & -I_{xy} & -I_{zx} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{zx} & -I_{yz} & I_z \end{bmatrix} \quad (24)$$

for the moment axis terms, i.e.,  $I_y$ , and the product of inertia terms, i.e.,  $I_{zx}$ .

TABLE 3.1.1-1

Equations of Motion

(first twelve system states)

$$\begin{aligned}
 \dot{\underline{r}}^{(I)} &= I_C^B \underline{\dot{v}}^{(B)} && \text{inertial mean of 50} \\
 \dot{\underline{v}}^{(B)} &= \frac{\rho v_A^2}{2m} \underline{c}_f + B_C^I \underline{g}^{(B)} (\underline{r}^{(I)}) - \underline{\omega} \times \underline{v}^{(B)} + \frac{f_T^{(B)}}{m} + \frac{f_P^{(B)}}{m} \\
 \dot{\underline{\theta}} &= \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \\
 I_{\underline{\omega}}^{(B)} &= \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = [I]^{-1} \left[ \frac{\rho v_A^2 A d}{2} \underline{c}_m + \frac{\rho v_A^2}{2} (\underline{r}_A - r_{cg})^{(B)} \times \underline{c}_f - \underline{\omega} \times ([I] \underline{\omega}) + \underline{T}_T + \underline{T}_P \right]
 \end{aligned}$$

The aerodynamic force and moment coefficients and plume forces are defined as functions of angle-of-attack,  $\alpha$ , and angle-of-sideslip,  $\beta$ , as shown in Figure 3.1.1-1. The body referenced relative velocity vector, removing the wind velocity,  $\underline{v}_w$ , from the vehicle velocity, is given by

$$\underline{v}_r = \underline{v}^{(B)} - B_{CI} \underline{v}_w = \underline{v}^{(B)} - B_{C_{LL}} \underline{v}_w^{(LL)} \quad (25)$$

where  $\underline{v}_w^{(LL)}$  is the local-level referenced wind velocity vector. The following equations define  $\alpha$  and  $\beta$  in terms of the components of  $\underline{v}_r$

$$\alpha = \tan^{-1} \left( \frac{v_{r3}}{v_{r1}} \right) \quad (26)$$

$$\beta = \sin^{-1} \left( \frac{v_{r2}}{v_m} \right) \quad (27)$$

where

$$v_m = (v_{r1}^2 + v_{r2}^2 + v_{r3}^2)^{1/2} \quad (28)$$

The resultant thrust force  $\underline{f}_T^{(B)}$  is expanded as

$$\underline{f}_T^{(B)} = \sum_{i=1}^n B_{Ci} \begin{bmatrix} f_{Ti}^{(Q)} \\ 0 \\ 0 \end{bmatrix} = \sum_{i=1}^n B_{Ci}^{(Q)} \underline{f}_{Ti}^{(Q)} \quad (29)$$

where the transformation matrix  $B_{Ci}^{(Q)}$  transforms the magnitude of thrust for each thrusting device,  $f_{Ti}^{(Q)}$ , from its center-line to the body coordinates. The general equation for  $f_{Ti}^{(Q)}$  is



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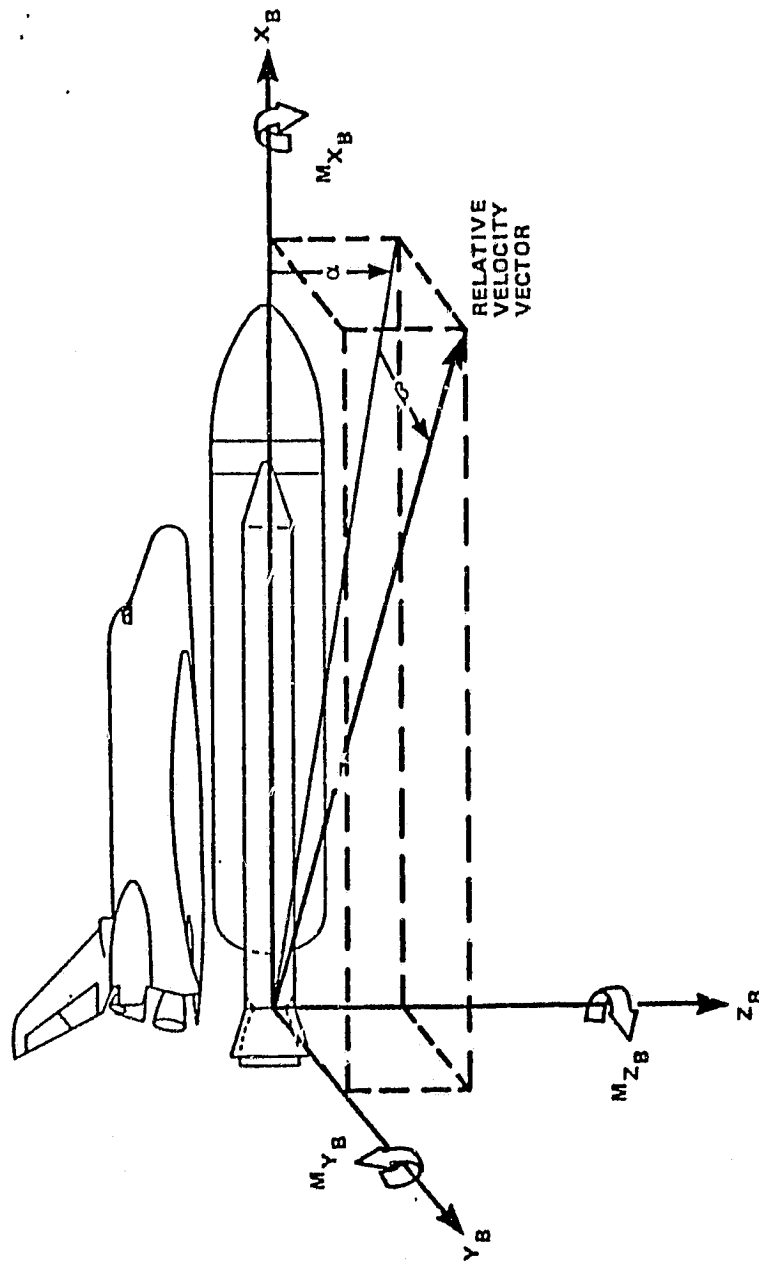


FIGURE 3.1.1-1. Body Referenced Wind Axis System

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$$f_{T_i} = f_{T_{ivac}} - P_{e_i} A_e$$

where

$$f_{T_{ivac}} = \text{vacuum thrust}$$

$$P_{e_i} = \text{motor exit pressure}$$

$$A_e = \text{motor exit cone area}$$

The matrix  $B_{C_i}^{Q_L}$  is different for the SSME's and SRB's and is given by

$$B_{C_i}^{Q_L} = \begin{matrix} B_{C_i}^{MP} & MP_{C_i}^{G} & G_{C_i}^{Q_L} & \text{SSME} \\ B_{C_i}^{Q_L} & & & \text{SRB} \end{matrix} \quad (30)$$

where

$B_{C_i}^{MP}$  = transformation from the engine mount plane to the body coordinates

$MP_{C_i}^{G}$  = transformation from the gimbal reference plane to mount plane (structural deformation)

$G_{C_i}^{Q_L}$  = transformation from enterline to the gimbal reference plane

$B_{C_i}^{Q_L}$  = transformation from SRB nozzle centerline to the body coordinates (gimbal angles).

The resultant thrust torque is the summation of the torque contribution from each thrusting device and is given by

$$\underline{T}_T^{(B)} = \sum_{i=1}^n (\underline{r}_{T_i}^{(B)} - \underline{r}_{cg}^{(B)}) \times \begin{bmatrix} B_{C_i} Q_i \\ f_{T_i} Q_i \\ 0 \\ 0 \end{bmatrix} \quad (31)$$

where

$\underline{r}_{T_i}^{(B)}$  = body coordinates of the thrust reference point for the  $i^{th}$  thrusting device.

### 3.1.2 Measurement Equations

The measurements assumed available for the filter/smoothing algorithm include inertial platform acceleration and attitudes, ground based radar tracking, SRB's head pressure, SSME's chamber pressures, liquid  $H_2$  flow rates, pressurant flow rates. The ET volumetric levels are available; however, due to their limited number (4), they may only be used for alternate checks of the filter/smoothing algorithm performance.

The propulsion related measurements will be treated in a separate section. In the following, the inertial platform acceleration measurements, attitude measurements and ground based tracking measurements models will be described for later linearization.

### 3.1.2.1 Platform Acceleration Measurements

Accelerometers mounted orthogonally on an inertially stabilized platform, not located at the vehicle center of gravity, sense externally applied special forces and accelerations due to body rotation. The accelerometer measurement is modeled by

$$\begin{aligned} \underline{a}_m^{(S)} = & S_C^P P_C^{P'} P'^C B \left[ \frac{\rho A v_m^2}{2m} \underline{c}_f + \frac{\underline{f}_T^{(B)}}{m} + \frac{\underline{f}_P^{(B)}}{m} \right. \\ & \left. + \underline{\omega} \times \underline{\omega} \times (\underline{r}_S^{(B)} - \underline{r}_{cg}^{(B)}) + \dot{\underline{\omega}} \times (\underline{r}_S^{(B)} - \underline{r}_{cg}^{(B)}) \right] + \underline{b}_a^{(S)} + \underline{v}_a^{(S)} \end{aligned} \quad (32)$$

where

$S_C^P$  = transformation from platform coordinates to sensing coordinates

$P_C^{P'}$  = transformation from misaligned platform coordinates to platform coordinates

$P'^C$  = transformation from body to misaligned platform coordinates

$\underline{r}_S^{(B)}$  = body coordinates of the platform center

$\underline{b}_a^{(S)}$  = accelerometer bias vector

$\underline{v}_a^{(S)}$  = accelerometer measurement noise vector

### 3.1.2.2 Platform Attitude Measurements

The inertially stabilized platform for the STS is a four axis IMU with a redundant roll axis [4]. Vehicle body attitudes are generated via quaternions [5]. It is assumed that an equivalent representation

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can be made to obtain vehicle attitude by a three rotation sequence of roll, pitch, yaw to transform from inertial to body coordinates. This approach has been used in reference [6].

The attitude angle measurement model is given by

$$\underline{\theta}_m^{(s)} = \underline{\theta} + \underline{b}_\theta^{(s)} + \underline{v}_\theta^{(s)} \quad (33)$$

where

$\underline{b}_\theta$  = platform misalignment bias vector (used to formulate  ${}^P_C P'$ )

$\underline{v}_\theta^{(s)}$  = attitude measurement noise vector.

The transformation matrix used to transform from body to inertial coordinates in terms of the elements of the  $\underline{\theta}$  vector is given by

$$I_{CB} = \begin{bmatrix} \cos\theta\cos\psi & \sin\psi\sin\theta\cos\phi & \cos\psi\sin\theta\cos\phi \\ & -\cos\psi\sin\phi & +\sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi & \cos\psi\sin\theta\sin\phi \\ & +\cos\psi\cos\phi & -\sin\psi\cos\phi \\ -\sin\theta & \sin\psi\cos\theta & \cos\psi\cos\theta \end{bmatrix} \quad (34)$$

### 3.1.2.3 Ground Based Tracking Measurements

Ground based radar tracking devices can provide measurements of range, azimuth and elevation from the radar sight to the vehicle. Azimuth and elevation are established relative to the sight's local level. If

the tracking device is a passive optical tracker (not laser) then only azimuth and elevation measurements are available requiring more than one to establish position information.

Defining  $x$ ,  $y$ , and  $z$  as the local east, north and up position of the vehicle relative to the ground based tracking device, the radar measurement equations are given by

$$\rho = (x^2 + y^2 + z^2)^{1/2} + b_\rho + v_\rho \quad (35)$$

$$A = \tan^{-1} \left( \frac{x}{y} \right) + b_A + v_A \quad (36)$$

$$E = \tan^{-1} \left( \frac{z}{\sqrt{x^2 + y^2}} \right) + b_E + \Delta_E + v_E \quad (37)$$

where

$b_\rho$ ,  $b_A$ ,  $b_E$  = range, azimuth, elevation biases

$\Delta_E$  = atmospheric refraction correction

$v_\rho$ ,  $v_A$ ,  $v_E$  = range, azimuth, elevation measurement noise.

The position vector of the vehicle relative to the tracking device is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \Delta \underline{r}_V^{(LL)} = LL_C^{ECF} [ECF_C^{ECI} \underline{r}^{(I)} - \underline{r}_{RDR}^{(ECF)}] \quad (38)$$

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where

$LL_C^{ECF}$  = transformation from earth center fixed to local level

$ECF_C^{ECI}$  = earth centered inertial to earth centered fixed

$\underline{r}_{RDR}^{(ECF)}$  = position vector of tracking device in ECF coordiantes.

The transformation matrix  $LL_C^{ECF}$  is given by

$$LL_C^{ECF} = \begin{bmatrix} -\sin\lambda & -\sin L \cos\lambda & \cos L \cos\lambda \\ \cos\lambda & -\sin L \sin\lambda & \cos L \sin\lambda \\ 0 & \cos L & \sin L \end{bmatrix} \quad (39)$$

where  $L$  and  $\lambda$  are the geodetic latitude and east longitude of the device. The transformation matrix  $ECF_C^{ECI}$  is given by

$$ECF_C^{ECI} = \begin{bmatrix} \cos[\omega_E(t - t_{RNP})] & \sin[\omega_E(t - t_{RNP})] & 0 \\ -\sin[\omega_E(t - t_{RNP})] & \cos[\omega_E(t - t_{RNP})] & 0 \\ 0 & 0 & 1 \end{bmatrix} [RNP] \quad (40)$$

where

$\omega_E$  = earth rotation rate

$t_{RNP}$  = time tag for RNP matrix

The position vector,  $\underline{r}_{RDR}^{(ECF)}$ , of the tracking device is given by

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$$\underline{r}_{\text{RDR}}^{(\text{ECF})} = \begin{bmatrix} \left( \frac{R_E}{\sqrt{\cos^2 L + (1 - e)^2 \sin^2 L}} + h \right) \cos L \cos \lambda \\ \left( \frac{R_E}{\sqrt{\cos^2 L + (1 - e)^2 \sin^2 L}} + h \right) \cos L \sin \lambda \\ \left( \frac{R_E (1 - e)^2}{\sqrt{\cos^2 L + (1 - e)^2 \sin^2 L}} + h \right) \sin L \end{bmatrix} \quad (41)$$

where

$R_E$  = equatorial radius of Fisher ellipsoid

$e$  = flattening of Fisher ellipsoid

$h$  = altitude of the device above Fisher ellipsoid



### 3.2 Linearized System State and Measurement Equations

The vehicle equations of motion are nonlinear functions of their motion variables and are implicit functions of other elements of the system states. The measurement equations involve similar function relationships. The linearizations for the filter/smoothing algorithm require partial derivatives with respect to the motion variables, i.e.,  $\underline{v}^{(B)}$  and  $\theta$ , and with respect to other elements of the state vector, yielding explicit functional relationships for the elements of interest.

For system state equations the partial derivatives will be presented in section 3.2.1 for the state elements in order of occurrence for the first twelve states. Other partial derivatives for candidate state elements will follow in section 3.3.1. The measurement equation partial derivatives for the first twelve states will be presented in section 3.2.2. Partial derivatives of the measurement equations for other candidate states will be presented in section 3.3.2.

The resulting partial derivatives are imbedded into the linearized system state matrix,  $F(\underline{x}(t), t)$ , as shown in Figure 3.2-1. A corresponding linearized measurement matrix,  $H(\underline{x}_k)$ , is similarly formed with the measurement equations' partial derivatives.

#### 3.2.1 System State Partial Derivatives

Partial derivatives of each of the equations listed in Table are developed in their order of occurrence with respect to the order of

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FIGURE 3.2-1. Linearization for Filter/Smoother Model  
(Dynamics)

	$\underline{r}^{(I)}$	$\underline{v}^{(B)}$	$\underline{\theta}$	$\underline{\omega}$
$\dot{\underline{r}}^{(I)}$	0	$\frac{\partial \dot{\underline{r}}^{(I)}}{\partial \underline{v}^{(B)}}$	$\frac{\partial \dot{\underline{r}}^{(I)}}{\partial \underline{\theta}}$	0
$\dot{\underline{v}}^{(B)}$	$\frac{\partial \dot{\underline{v}}^{(B)}}{\partial \underline{r}^{(I)}}$	$\frac{\partial \dot{\underline{v}}^{(B)}}{\partial \underline{v}^{(B)}}$	$\frac{\partial \dot{\underline{v}}^{(B)}}{\partial \underline{\theta}}$	0
$\dot{\underline{\theta}}$	0	0	$\frac{\partial \dot{\underline{\theta}}}{\partial \underline{\theta}}$	$\frac{\partial \dot{\underline{\theta}}}{\partial \underline{\omega}}$
$\dot{\underline{\omega}}$	$\frac{\partial \dot{\underline{\omega}}}{\partial \underline{r}^{(I)}}$	$\frac{\partial \dot{\underline{\omega}}}{\partial \underline{v}^{(B)}}$	$\frac{\partial \dot{\underline{\omega}}}{\partial \underline{\theta}}$	$\frac{\partial \dot{\underline{\omega}}}{\partial \underline{\omega}}$

the corresponding states. Partial derivatives of thrust terms are presented as though for a single device.

#### Inertial Position Rate Equation

The first nonzero partial derivative of the  $\dot{\underline{r}}^{(I)}$  equation is with respect to  $\underline{v}^{(B)}$ :

$$\frac{\partial}{\partial \underline{v}^{(B)}} (\dot{\underline{r}}^{(I)}) = I_C^B. \quad (42)$$

The second nonzero partial derivative is with respect to  $\underline{\theta}$ . This partial derivative results in a third order tensor and occurs frequently in later developments. The generalized form is presented in Appendix A.

#### Body Velocity Rate Equation

The partial derivative of  $\dot{\underline{v}}^{(B)}$  with respect to  $\underline{r}^{(I)}$  is given for altitude terms approximately as

$$\frac{\partial}{\partial \underline{r}^{(I)}} \dot{\underline{v}}^{(B)} = \frac{\partial}{\partial h} \frac{\underline{r}^{(I)T}}{|\underline{r}^{(I)}|} \quad (43)$$

where

$$\begin{aligned} \frac{\partial \dot{\underline{v}}^{(B)}}{\partial h} = & \frac{A v_m^2}{\partial m} \frac{\partial p}{\partial h} c_f + \frac{\rho A v_m}{m} c_f \frac{\partial v_m}{\partial h} + \frac{\rho A v_m^2}{2m} \frac{\partial c_f}{\partial \alpha} \frac{\partial \alpha}{\partial h} + \frac{\rho A v_m^2}{2m} \frac{\partial c_f}{\partial \beta} \frac{\partial \beta}{\partial h} \\ & + \frac{1}{m} ({}^B C_L \frac{\partial f_T^{(L)}}{\partial p_s} \frac{\partial p_s}{\partial h} + \frac{\partial f_p^{(B)}}{\partial h} + \frac{\partial f_p^{(B)}}{\partial h} \frac{\partial \alpha}{\partial h} + \frac{\partial f_p^{(B)}}{\partial \beta} \frac{\partial \beta}{\partial h}) \end{aligned} \quad (44)$$

The partial derivatives of  $\frac{\partial v_m}{\partial v^{(B)}}, \frac{\partial \alpha}{\partial v^{(B)}}, \frac{\partial \beta}{\partial v^{(B)}}, \frac{\partial \alpha}{\partial h}, \frac{\partial \beta}{\partial h}$  and  $\frac{\partial v_m}{\partial h}$

occur frequently and are given in Appendix B.

The gravity vector  ${}^{B_C I} \underline{g}^{(I)}(\underline{r}^{(I)})$  partial derivative with respect to  $\underline{r}^{(I)}$  is

$${}^{B_C I} \frac{\partial \underline{g}^{(I)}(\underline{r}^{(I)})}{\partial \underline{r}^{(I)}} = {}^{B_C I} \frac{\mu}{|\underline{r}^{(I)}|^3} \begin{bmatrix} \frac{r_1^2}{|\underline{r}|^2} - 1 & \frac{r_1 r_2}{|\underline{r}|^2} & \frac{r_1 r_2^3}{|\underline{r}|^2} \\ \frac{r_1 r_2}{|\underline{r}|^2} & \frac{r_2^2}{|\underline{r}|^2} - 1 & \frac{r_2 r_3}{|\underline{r}|^2} \\ \frac{r_1 r_3}{|\underline{r}|^2} & \frac{r_2 r_3}{|\underline{r}|^2} & \frac{r_3^2}{|\underline{r}|^2} - 1 \end{bmatrix} \quad (45)$$

where

$\mu$  = gravitational constant.

The partial derivative,  $\frac{\partial \dot{\underline{v}}^{(B)}}{\partial \underline{r}^{(I)}}$ , is the sum of the matrices in equations

and .

The partial derivative of  $\dot{\underline{v}}^{(B)}$  with respect to  $\underline{v}^{(B)}$  is given by

$$\begin{aligned} \frac{\partial \dot{\underline{v}}^{(B)}}{\partial \underline{v}^{(B)}} &= \frac{\rho A v_m}{m} c_f \frac{\partial v_m}{\partial v^{(B)}} + \frac{\rho A v_m^2}{2m} \frac{c_f}{\partial \alpha} \frac{\partial \alpha}{\partial v^{(B)}} + \frac{\rho A v_m^2}{2m} \frac{\partial c_f}{\partial \beta} \frac{\partial \beta}{\partial v^{(B)}} \\ &+ \frac{1}{m} \left[ \frac{\partial f_p}{\partial \alpha} \frac{\partial \alpha}{\partial v^{(B)}} + \frac{\partial f_p}{\partial \beta} \frac{\partial \beta}{\partial v^{(B)}} \right] - \{\underline{\omega}\} \end{aligned} \quad (46)$$

where

$\{\underline{\omega}\}$  = skew symmetric matrix made from the elements of the vector  $\underline{\omega}$   
and equivalent to the cross product operator  $\underline{\omega} \times ( )$ .

The partial derivative of  $\dot{\underline{v}}^{(B)}$  with respect to  $\underline{\theta}$  is

$$\begin{aligned} \frac{\partial \dot{\underline{v}}^{(B)}}{\partial \underline{\theta}} = & \frac{\rho A v_m}{m} \underline{c}_f \frac{\partial v_m}{\partial v_r} \frac{\partial v_r}{\partial \underline{\theta}} + \frac{\rho A v_m^2}{2m} \frac{\partial \underline{c}_f}{\partial \alpha} \frac{\partial \alpha}{\partial v_r} \frac{\partial v_r}{\partial \underline{\theta}} \\ & + \frac{\rho A v_m^2}{2m} \frac{\partial \underline{c}_f}{\partial \beta} \frac{\partial \beta}{\partial v_r} \frac{\partial v_r}{\partial \underline{\theta}} + \frac{\partial}{\partial \underline{\theta}} [{}^B C^I \underline{g}^{(I)}(\underline{r}^{(I)})] \\ & + \frac{1}{m} \left[ \frac{\partial f_p^{(B)}}{\partial \alpha} \frac{\partial \alpha}{\partial v_r} \frac{\partial v_r}{\partial \underline{\theta}} + \frac{\partial f_p^{(B)}}{\partial \beta} \frac{\partial \beta}{\partial v_r} \frac{\partial v_r}{\partial \underline{\theta}} \right]. \end{aligned} \quad (47)$$

The partial derivatives of  $\frac{\partial}{\partial v_r}$  are given in Appendix B and the partial

derivative  $\frac{\partial v_r}{\partial \underline{\theta}}$  is given in Appendix A. The last partial derivative

is given in Appendix C.

The partial derivative of  $\dot{\underline{v}}^{(B)}$  with respect to  $\underline{\omega}$  is

$$\frac{\partial \dot{\underline{v}}^{(B)}}{\partial \underline{\omega}} = \frac{\rho A v_m^2}{2m} \frac{\partial \underline{c}_f}{\partial \underline{\omega}} + \{\underline{v}\}^{(B)} \quad (48)$$

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### Euler Angle Rate Equation

The Euler angle rate equation is a function of both the Euler angles and the inertial rates. The linearization will yield the two associated matrices.

First with respect to the vector  $\underline{\theta}$ , the following matrix results

$$\frac{\partial \dot{\underline{\theta}}}{\partial \underline{\theta}} = \begin{bmatrix} q \cos \phi \tan \theta - r \sin \phi \tan \theta & q \sin \phi \sec^2 \theta + r \cos \phi \sec^2 \theta & 0 \\ -q \sin \phi - r \cos \phi & 0 & 0 \\ q \cos \phi \sec \theta - r \sin \phi \cos \theta & q \sin \phi \sec \theta \tan \theta + r \cos \phi \sec \theta \tan \theta & 0 \end{bmatrix} \quad (49)$$

The partial derivative of  $\dot{\underline{\theta}}$  with respect to  $\underline{\omega}$  is

$$\frac{\partial \dot{\underline{\theta}}}{\partial \underline{\omega}} = \begin{bmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{bmatrix} \quad (50)$$

### Inertial Angular Acceleration Equation

The first partial derivative of this equation is with respect to the vector  $\underline{r}^{(I)}$ . Using the approximation indicated in equation , this partial derivative is

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$$\begin{aligned}
 \frac{\partial \dot{\underline{\omega}}}{\partial \underline{r}^{(I)}} = & [\underline{I}]^{-1} \left\{ \frac{\text{Adv}_m^2}{2} \frac{\partial \rho}{\partial h} \underline{c}_m + (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \rho v_m \underline{Ac}_f \right\} \frac{\partial v_m}{\partial h} \\
 & + \left( \frac{\rho v_m^2 \text{Ad}}{2} \frac{\partial \underline{c}_m}{\partial \alpha} + (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \frac{\rho v_m^2 \text{Ad}}{2} \frac{\partial \underline{c}_f}{\partial \alpha} \right) \frac{\partial \alpha}{\partial h} \\
 & + \left( \frac{\rho v_m^2 \text{Ad}}{2} \frac{\partial \underline{c}_m}{\partial \beta} + (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \frac{\rho v_m^2 \text{Ad}}{2} \frac{\partial \underline{c}_f}{\partial \beta} \right) \frac{\partial \beta}{\partial h} \\
 & + \left( (\underline{r}_T^{(B)} - \underline{r}_{cg}^{(B)}) \times \text{B}_C \underline{Q}_L \frac{\partial f^{(Q)}}{\partial p_s} \frac{\partial p_s}{\partial h} + \frac{\partial T_p}{\partial h} + \frac{\partial T_p}{\partial \alpha} \frac{\partial \alpha}{\partial h} + \frac{\partial T_p}{\partial \beta} \frac{\partial \beta}{\partial h} \right) \frac{\underline{r}^{(I)T}}{|\underline{r}^{(I)}|}
 \end{aligned}
 \tag{51}$$

Next, with respect to the vector  $\underline{v}^{(B)}$ , the partial derivative is

$$\begin{aligned}
 \frac{\partial \dot{\underline{\omega}}}{\partial \underline{v}^{(B)}} = & [\underline{I}]^{-1} \left\{ (\rho v_m \text{Ad} \underline{c}_m + (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \rho v_m \underline{Ac}_f) \frac{\partial v_m}{\partial \underline{v}^{(B)}} \right. \\
 & + \left( \frac{\rho v_m^2 \text{Ad}}{2} \frac{\partial \underline{c}_m}{\partial \alpha} + (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \frac{\rho v_m^2 \text{Ad}}{2} \frac{\partial \underline{c}_f}{\partial \alpha} \right) \frac{\partial \alpha}{\partial \underline{v}^{(B)}} \\
 & + \left( \frac{\rho v_m^2 \text{Ad}}{2} \frac{\partial \underline{c}_m}{\partial \beta} + (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \frac{\rho v_m^2 \text{Ad}}{2} \frac{\partial \underline{c}_f}{\partial \beta} \right) \frac{\partial \beta}{\partial \underline{v}^{(B)}} \\
 & \left. + \frac{\partial T_p}{\partial \alpha} \frac{\partial \alpha}{\partial \underline{v}^{(B)}} + \frac{\partial T_p}{\partial \beta} \frac{\partial \beta}{\partial \underline{v}^{(B)}} \right\} .
 \end{aligned}
 \tag{52}$$

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The partial derivative with respect to the vector  $\underline{\theta}$  is

$$\begin{aligned}
 \frac{\partial \underline{\dot{\omega}}}{\partial \underline{\theta}} &= [\underline{I}]^{-1} \{ (\rho v_m^2 \underline{A} \underline{c}_m + (\underline{r}_A^{(B)} - \underline{c}_{cg}^{(B)}) \times \rho v_m^2 \underline{A} \underline{c}_f) \frac{\partial v_m}{\partial \underline{\theta}} \\
 &+ \left( \frac{\rho v_m^2 \underline{A}}{2} \frac{\partial \underline{c}_m}{\partial \alpha} + (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \frac{\rho v_m^2 \underline{A}}{2} \frac{\partial \underline{c}_f}{\partial \alpha} \right) \frac{\partial \alpha}{\partial \underline{\theta}} \\
 &+ \left( \frac{\rho v_m^2 \underline{A}}{2} \frac{\partial \underline{c}_m}{\partial \beta} + (\underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)}) \times \frac{\rho v_m^2 \underline{A}}{2} \frac{\partial \underline{c}_f}{\partial \beta} \right) \frac{\partial \beta}{\partial \underline{\theta}} \\
 &+ \frac{\partial \underline{T}_p}{\partial \alpha} \frac{\partial \alpha}{\partial \underline{\theta}} + \frac{\partial \underline{T}_p}{\partial \beta} \frac{\partial \beta}{\partial \underline{\theta}} \}.
 \end{aligned}
 \tag{53}$$

The final partial derivative for the first twelve states is with respect to the vector  $\underline{\omega}$ . This operation yields

$$\begin{aligned}
 \frac{\partial \underline{\dot{\omega}}}{\partial \underline{\omega}} &= [\underline{I}]^{-1} \left[ \frac{\rho A v_m^2}{2} \frac{\partial \underline{c}_m}{\partial \underline{\omega}} + \{ \underline{r}_A^{(B)} - \underline{r}_{cg}^{(B)} \} \frac{\rho A v_m^2}{2} \frac{\partial \underline{c}_f}{\partial \underline{\omega}} \right. \\
 &\left. + \{ \underline{I} \underline{\omega} \} - \{ \underline{\omega} \} \underline{I} \right]
 \end{aligned}
 \tag{54}$$



### 3.2.2 Measurement Partial Derivatives

The measurements assumed to be available, as discussed earlier, include ground based radar tracking, inertially stabilized platform attitudes relative to the vehicle body, and stabilized platform mounted 3 axis orthogonal accelerations. As with the state dynamics matrix, the measurement equations are linearized about the best state estimates.

#### Radar Track Measurement Equation

Referring to the radar track measurement equations, the required partial derivatives are

$$\frac{\partial \rho}{\partial \underline{r}^{(I)}} = \frac{\partial \rho}{\partial \Delta \underline{r}_{-v}^{(LL)}} \frac{\partial \Delta \underline{r}_{-v}^{(LL)}}{\partial \underline{r}^{(I)}} \quad (55)$$

$$\frac{\partial A}{\partial \underline{r}^{(I)}} = \frac{\partial A}{\partial \Delta \underline{r}_{-v}^{(LL)}} \frac{\partial \Delta \underline{r}_{-v}^{(LL)}}{\partial \underline{r}^{(I)}} \quad (56)$$

$$\frac{\partial E}{\partial \underline{r}^{(I)}} = \frac{\partial E}{\partial \Delta \underline{r}_{-v}^{(LL)}} \frac{\partial \Delta \underline{r}_{-v}^{(LL)}}{\partial \underline{r}^{(I)}} \quad (57)$$

The last partial derivative in each of these equations,  $\frac{\partial \Delta \underline{r}_{-v}^{(LL)}}{\partial \underline{r}^{(I)}}$ , is

$$\frac{\partial \Delta \underline{r}_{-v}^{(LL)}}{\partial \underline{r}^{(I)}} = {}^{LL}_C ECF {}^{ECF}_C ECI \quad (58)$$

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The rest of the required partial derivatives are

$$\frac{\partial \rho}{\partial \Delta \underline{r}_{-v}^{(LL)}} = \Delta \underline{r}_{-v}^{(LL)T} / |\Delta \underline{r}_{-v}| \quad (59)$$

$$\frac{\partial A}{\partial \Delta \underline{r}_{-v}^{(LL)}} = \left[ \frac{y}{x^2 + y^2}, \quad \frac{-x}{x^2 + y^2}, \quad 0 \right] \quad (60)$$

$$\frac{\partial E}{\partial \Delta \underline{r}_{-v}^{(LL)}} = \left[ \frac{-xz}{\rho^2 \sqrt{x^2 + y^2}}, \quad \frac{-yz}{\rho^2 \sqrt{x^2 + y^2}}, \quad \frac{\sqrt{x^2 + y^2}}{\rho^2} \right] \quad (61)$$

#### Inertially Stabilized Platform Attitude Equation

The inertial platform is assumed to provide attitude angle measurements of the true attitude plus an attitude bias plus measurement noise. The partial derivative of the measured attitudes with respect to the vector  $\underline{\theta}$  yields an identity matrix.

#### Accelerometer Measurement Equation

The accelerometer senses specific body forces excluding gravity along the sensing axes. With reference to the accelerometer equation, the partial derivative with respect to  $\underline{r}^{(I)}$  is

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$$\begin{aligned}
 \frac{\partial \underline{a}_m^{(S)}}{\partial \underline{r}^{(I)}} = & S_C^B \left[ \frac{\rho A v_m^2}{2m} \frac{\partial \rho}{\partial h} \underline{c}_f + \frac{\rho A v_m}{m} \underline{c}_f \frac{\partial v_m}{\partial h} \right. \\
 & + \frac{\rho A v_m^2}{2m} \frac{\partial \underline{c}_f}{\partial \alpha} \frac{\partial \alpha}{\partial h} + \frac{\rho A v_m^2}{2m} \frac{\partial \underline{c}_f}{\partial \beta} \frac{\partial \beta}{\partial h} \\
 & \left. + \frac{1}{m} \left( {}^B C^L \frac{\partial f^{(L)}}{\partial \rho_s} \frac{\partial p_s}{\partial h} + \frac{\partial f^{(B)}}{\partial \alpha} \frac{\partial \alpha}{\partial h} + \frac{\partial f^{(B)}}{\partial \beta} \frac{\partial \beta}{\partial h} \right) \right] \frac{\underline{r}^{(I)T}}{|\underline{r}^{(I)}|}
 \end{aligned} \tag{62}$$

The partial derivative with respect to  $\underline{v}^{(B)}$  yields

$$\begin{aligned}
 \frac{\partial \underline{a}_m^{(S)}}{\partial \underline{v}^{(B)}} = & S_C^B \left[ \frac{\rho A v_m}{m} \underline{c}_f \frac{\partial v_m}{\partial \underline{v}^{(B)}} + \frac{\rho A v_m^2}{2m} \frac{\partial \underline{c}_f}{\partial \alpha} \frac{\partial \alpha}{\partial \underline{v}^{(B)}} + \frac{\rho A v_m^2}{2m} \frac{\partial \underline{c}_f}{\partial \beta} \frac{\partial \beta}{\partial \underline{v}^{(B)}} \right. \\
 & \left. + \frac{1}{m} \left[ \frac{\partial f^{(B)}}{\partial \alpha} \frac{\partial \alpha}{\partial \underline{v}^{(B)}} + \frac{\partial f^{(B)}}{\partial \beta} \frac{\partial \beta}{\partial \underline{v}^{(B)}} \right] \right]
 \end{aligned} \tag{63}$$

For the partial derivative of the accelerometer with respect to the vector  $\underline{\theta}$ , the measurement equation is temporarily rewritten as

$$\underline{a}_m^{(S)} = S_C^{P'} P' C^B \underline{s}^{(B)} + \underline{b}_a^{(S)} \tag{64}$$

where the vector  $\underline{s}^{(B)}$  represents the sum of the aerodynamic, thrust, plume and rotational coupling terms. The matrix  $P' C^B$  is the same matrix as  $I_C^B$ . The required partial derivative results from

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$$\frac{\partial \underline{a}_m^{(S)}}{\partial \underline{\theta}} = S_C^{P'} \frac{\partial}{\partial \underline{\theta}} I_C^B \underline{s}^{(B)}. \quad (65)$$

The partial derivative on the right hand side is developed in Appendix A with the vector  $\underline{s}^{(B)}$  representing the sum of the terms indicated above.

The final partial derivative for the accelerometer measurement is with respect to the body rotation vector  $\underline{\omega}$ . Defining

$$\Delta \underline{r}_s = \begin{bmatrix} \Delta r_{1s} \\ \Delta r_{2s} \\ \Delta r_{3s} \end{bmatrix} = (\underline{r}_s^{(B)} - \underline{r}_{cg}^{(B)}) \quad (66)$$

and denoting  $\omega_i$  as the  $i^{th}$  element of the vector  $\underline{\omega}$ , the resulting matrix is

$$\frac{\partial \underline{a}_m}{\partial \underline{\omega}} = S_C^B \begin{bmatrix} \omega_2 \Delta r_2 + \omega_3 \Delta r_3 & \omega_1 \Delta r_2 - 2\omega_2 \Delta r_1 & \omega_1 \Delta r_3 - 2\omega_3 \Delta r_1 \\ \omega_2 \Delta r_1 - 2\omega_1 \Delta r_2 & \omega_1 \Delta r_1 + \omega_3 \Delta r_3 & \omega_2 \Delta r_3 - 2\omega_3 \Delta r_2 \\ \omega_3 \Delta r_1 - 2\omega_1 \Delta r_3 & \omega_3 \Delta r_2 - 2\omega_2 \Delta r_3 & \omega_1 \Delta r_1 + \omega_2 \Delta r_2 \end{bmatrix} \quad (67)$$

### 3.3 Propulsion Parameter States and Measurements

A candidate approach for incorporating the NASA propulsion model's capabilities has been identified. This approach utilizes nominal predicted values of thrust, pressure, propellant and pressurant mass flow rates, and utilizes sensitivities or partial derivatives of these variables with respect to the independent parameters selected for estimation by the algorithm.

The approach is to include deviations from nominal values of measured chamber pressure, power level, propellant and pressurant mass flow rates as states. The models assumed for these deviations are time correlated random processes. Then as states, partial derivatives of the first twelve states with respect to these variables will be required.

For the SSME and SRB, this modeling approach is discussed in the following. Additionally, the necessary partial derivatives of the first twelve state variables with respect to the additional states are presented.

#### 3.3.1 SSME Propulsion Parameter Model

For the SSME, the total actual values of vacuum thrust and oxidizer mass flow rates are modeled by

$$f_T = f_{T_{nom}} + \Delta f_T \quad (68)$$

and

$$\dot{m}_{O_2} = \dot{m}_{O_2_{nom}} + \Delta \dot{m}_{O_2} \quad (69)$$

The measurements of fuel mass flow rate, pressurant mass flow rates and power level are modeled as

$$\dot{m}_{H_2} = \dot{m}_{H_2_{nom}} + \Delta \dot{m}_{H_2} + b_{\dot{m}_{H_2}} + s_{\dot{m}_{H_2}} \quad (70)$$

$$\dot{m}_{H_2_p} = \dot{m}_{H_2_{p_{nom}}} + \Delta \dot{m}_{H_2_p} + b_{\dot{m}_{H_2_p}} + s_{\dot{m}_{H_2_p}} \quad (71)$$

$$\dot{m}_{O_2_p} = \dot{m}_{O_2_{p_{nom}}} + \Delta \dot{m}_{O_2_p} + b_{\dot{m}_{O_2_p}} + s_{\dot{m}_{O_2_p}} \quad (72)$$

and

$$PL = PL_{nom} + \Delta PL + b_{PL} + s_{PL} \quad (73)$$

These measured quantities include measurement noise  $s_{( )}$  and potential bias states  $b_{( )}$  modeled as random constants. In these measurements, the  $\Delta$ 'd variables are to be included as states in the estimation algorithm. If the nominal values are zero or unknown, then the  $\Delta$ 'd variables absorb the entire estimate. Where required, the estimate for the variables used in the estimation algorithm is formed using the nominal and the estimate of the deviation, etc. In example, thrust and fuel mass flow rate estimates are formed as

$$\hat{f}_T = f_{T_{nom}} + \hat{\Delta f}_T \quad (74)$$

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and

$$\dot{\hat{m}}_{H_2} = \dot{m}_{H_2 \text{ nom}} + \Delta \dot{m}_{H_2} + \hat{b}_{\dot{m}_{H_2}} \quad (75)$$

The deviation or  $\Delta$ 'd measurement variables are modeled as time correlated random variables. This permits these variables to vary within a band of frequencies. The typical model is then given as

$$\frac{d}{dt} \Delta( ) = -\frac{1}{T( )} \Delta( ) + \frac{1}{T( )} s( ) \quad (76)$$

where the parenthesis ( ) would be replaced by the variables, i.e.,  $\dot{m}_{H_2}$ . For the SSME, an additional variable  $\Delta c^*$  is modeled as in equation 76 and included as a state mult variable with the  $\Delta$ 'd measurement variables.

The thrust deviation is expanded as in the following truncated Taylor series as a function of the independent parameters.

$$\begin{aligned} \Delta f_T = & \frac{\partial f_T}{\partial \dot{m}_{H_2 p}} \Delta \dot{m}_{H_2 p} + \frac{\partial f_T}{\partial \dot{m}_{O_2 p}} \Delta \dot{m}_{O_2 p} + \frac{\partial f_T}{\partial \Delta c^*} \Delta c^* \\ & + \frac{\partial f_T}{\partial PL} \Delta PL + \frac{\partial f_T}{\partial MR} \Delta MR. \end{aligned} \quad (77)$$

In the  $\dot{\underline{v}}^{(B)}$  and  $\dot{\underline{\omega}}$  equations, with equation 77 replacing  $f_{T_i}$ , the partial derivatives of  $f_T$  with respect to the  $\Delta$ 'd variables are obtained directly from equation 77.

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It is desirable to include vehicle mass as a state. The SSME's system contribution to the mass deviation is given by

$$\dot{\Delta m}_{SSME's} = \sum_i (\dot{\Delta m}_{H_2 i} + \dot{\Delta m}_{O_2 i} - \dot{\Delta m}_{H_2 p_i} - \dot{\Delta m}_{O_2 p_i}). \quad (78)$$

In equation 78, the  $\dot{\Delta m}_{O_2}$  contribution to the mass deviation is not available from measurements. As with the thrust deviation, this quantity is formed as

$$\begin{aligned} \dot{\Delta m}_{O_2} = & \frac{\partial \dot{m}_{O_2}}{\partial \dot{m}_{H_2 p}} \dot{\Delta m}_{H_2 p} + \frac{\partial \dot{m}_{O_2}}{\partial \dot{m}_{O_2 p}} \dot{\Delta m}_{O_2 p} + \frac{\partial \dot{m}_{O_2}}{\partial \Delta c^*} \Delta c^* \\ & + \frac{\partial \dot{m}_{O_2}}{\partial PL} \Delta PL + \frac{\partial \dot{m}_{O_2}}{\partial MR} \Delta MR. \end{aligned} \quad (79)$$

which is in terms of other estimated state variables. In equations 77 and 79, the deviation in mixture ratio,  $\Delta MR$ , is obtained algebraically from

$$\Delta MR = \frac{\dot{m}_{H_2 nom} - \dot{m}_{H_2}}{\frac{\partial \dot{m}_{H_2}}{\partial MR}} \quad (80)$$



### 3.3.2 SRB Propulsion Parameter Model

The approach for the SRB modeling follows closely that used for the SSME. Candidate independent parameters include propellant burn rate exponent,  $a$ , and motor efficiency coefficient,  $c_m$ . Others can be added using this technique.

The actual value of vacuum thrust is given by equation 68. The only measurement available for the SRB is the total pressure at the forward head end of the motor case and is modeled as

$$P_{O_H} = P_{O_{H_{nom}}} + \Delta P_{O_H} + b_{P_{O_H}} + s_{P_{O_H}} \quad (81)$$

where  $b_{P_{O_H}}$  and  $s_{P_{O_H}}$  represents a bias and measurement noise respectively.

The independent parameters,  $\Delta a$  and  $\Delta c_m$ , are included in the model as states. The model assumed can be as given by equation 76 or another suitable dynamical process, i.e., random constant.

The thrust deviation is given by the following truncated Taylor series as a function of the candidate independent parameters.

$$\Delta f_T = \frac{\partial T}{\partial a} \Delta a + \frac{\partial T}{\partial c_m} \Delta c_m + \dots \quad (82)$$

The partial derivatives for the  $\dot{\underline{v}}^{(B)}$  and  $\dot{\underline{\omega}}$  equations with respect to the independent parameters are obtained directly from equation 82. The mass deviation equation for the SRB is given as

$$\dot{\Delta m}_{SRB} = \sum_i (\dot{\Delta m}_i) \quad (83)$$

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where

$$\dot{\Delta m}_i \approx \frac{\partial \dot{m}}{\partial a} \Delta a + \dots \quad (84)$$

The head pressure deviation,  $\Delta P_{OH}$ , is expanded similarly

$$\Delta P_{OH} = \frac{\partial P_{OH}}{\partial a} \Delta a + \dots \quad (85)$$

### 3.3.3 Vehicle Mass State Variable

The total rate of change of vehicle mass is given by

$$\frac{d}{dt}(m) = \dot{m}_{SSME_{nom}} + \dot{m}_{SRB_{nom}} + \dot{\Delta m}_{SSME} + \dot{\Delta m}_{SRB} + \dot{m}_{NON-CONSUME} \quad (86)$$

The first two terms in this equation are the apriori assumed nominal values. The third and fourth terms were discussed earlier. The last term should be zero.

The equations, state and measurement, in which mass occurs are the  $\dot{\underline{v}}^{(B)}$  and  $\underline{a}_{\underline{m}}$  equations. Assuming equation 86 can be summarized as  $\dot{m} + \sum_i \dot{\Delta m}_i$  then the mass can be written as  $m + \sum_i \Delta m_i$ . Replacing this expression for the mass in the two indicated equations yields the following partial derivatives with respect to the individual  $\Delta m_i$ 's

$$\frac{\partial \underline{v}^{(B)}}{\partial \Delta m_i} = - \frac{1}{(m + \sum_i \Delta m_i)^2} \left( \frac{\rho v_m^2 A}{2} c_f + \underline{f}_p^{(B)} + B_C^Q \underline{f}_T^{(Q)} \right) \quad (87)$$

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and

$$\frac{\partial a_m^{(S)}}{\partial \Delta m_i} = - \frac{1}{(m + \sum_i \Delta m_i)^2} S_C^B \left( \frac{\rho v_m^2 A}{2} c_f + \frac{f_p^{(B)}}{p} + E_C^Q \frac{f_T^{(Q)}}{T_i} \right) \quad (88)$$

#### 4.0 PROJECTED ACTIVITIES DURING UPCOMING MONTH

The mathematical development of the system and measurement equations and their linearization will be concluded. Included will be partial derivatives with respect to  $\underline{r}_{cg}$ ,  $\underline{v}_w$ , and  $I$  for center-of-gravity, wind velocity, and moment of inertia uncertainties. Additionally, partial derivatives with respect to platform tilt errors in the accelerometer measurement equations will be formulated.

The estimation program development will be initiated. The program will be structured to permit additional parameter states to be included with minimal impact to the program. As experience is gained with the operation and results of the program, changes will inevitably be required. Initial checkouts will focus on the propulsion parameter estimation with other parameters assumed known, i.e., aerodynamic coefficients. These will be included in the state model later for a more extensive evaluation.

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# APPENDIX A

## PARTIAL DERIVATIVE OF THE VECTOR $I_C^B \underline{v}$ wrt $\underline{\theta}$

This partial derivative is one of several that occurs frequently in the formulation of the linearized system state and measurement equations. The desired partial derivative is

$$\frac{\partial}{\partial \underline{\theta}} \begin{bmatrix} (\cos\theta\cos\phi)v_1 + (\sin\phi\sin\theta\cos\psi)v_2 + (\cos\phi\sin\theta\cos\psi)v_3 \\ (\cos\theta\sin\phi)v_1 + (\sin\phi\sin\theta\sin\psi)v_2 + (\cos\phi\sin\theta\sin\psi)v_3 \\ (-\sin\theta)v_1 + (\sin\phi\cos\theta)v_2 + (\cos\phi\cos\theta)v_3 \end{bmatrix} \quad \text{A-1}$$

The resulting matrix is given in Table A-1.

TABLE A-1. Partial Derivative of  $I_C^B$  v wrt  $\theta$

$-\sin\theta\cos\phi \ v_1$	$-\cos\theta\sin\phi \ v_1$	
$+\sin\phi\cos\theta\cos\phi \ v_2$	$-(\sin\phi\sin\theta\sin\phi + \cos\phi\cos\phi)v_2$	$(\cos\phi\sin\theta\cos\phi + \sin\phi\sin\phi)v_2$
$+\cos\phi\cos\theta\cos\phi \ v_3$	$-(\cos\phi\sin\theta\sin\phi - \sin\phi\cos\phi)v_3$	$-(\sin\phi\sin\theta\cos\phi - \cos\phi\sin\phi)v_3$
$-\sin\theta\sin\phi \ v_1$	$\cos\theta\cos\phi \ v_1$	
$+\sin\phi\cos\theta\sin\phi \ v_2$	$+(\sin\phi\sin\theta\cos\phi - \cos\phi\sin\phi)v_2$	$(\cos\phi\sin\theta\sin\phi - \sin\phi\cos\phi)v_2$
$+\cos\phi\cos\theta\sin\phi \ v_3$	$+(\cos\phi\sin\theta\cos\phi + \sin\phi\sin\phi)v_3$	$-(\sin\phi\sin\theta\sin\phi + \cos\phi\cos\phi)v_3$
$-\cos\theta \ v_1$		
$-\sin\phi\sin\theta \ v_2$	0	$\cos\phi\cos\theta \ v_2$
$-\cos\phi\sin\theta \ v_3$		$-\sin\phi\cos\theta \ v_3$

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APPENDIX B

$$\frac{\partial v_m}{\partial v^{(B)}}, \frac{\partial \alpha}{\partial v^{(B)}}, \frac{\partial \beta}{\partial v^{(B)}}, \frac{\partial \alpha}{\partial h}, \frac{\partial \beta}{\partial h} \text{ and } \frac{\partial v_m}{\partial h} \text{ Expressions}$$

These partial derivatives occur frequently and will be developed in this appendix. The equation for  $\underline{v_r}$  is

$$\underline{v_r} = \underline{v}^{(B)} - B_{C LL} \underline{v_w}^{(LL)} \quad B-1$$

Since the wind velocity,  $\underline{v_w}^{(LL)}$ , is only a function of altitude then

$$\frac{\partial}{\partial \underline{v_r}} = \frac{\partial}{\partial \underline{v}^{(B)}} \quad B-2$$

The first partial derivative,  $\frac{\partial v_m}{\partial \underline{v}^{(B)}}$ , is

$$\frac{\partial v_m}{\partial \underline{v}^{(B)}} = \frac{1}{v_m} \begin{bmatrix} v_{r1} \\ v_{r2} \\ v_{r3} \end{bmatrix} \quad B-3$$

The second,  $\frac{\partial \alpha}{\partial \underline{v}^{(B)}}$ , is given by

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$$\frac{\partial \alpha}{\partial \underline{v}^{(B)}} = \begin{bmatrix} \frac{-v_{r_3}}{v_{r_1}^2 + v_{r_3}^2} \\ 0 \\ \frac{v_{r_1}}{v_{r_1}^2 + v_{r_3}^2} \end{bmatrix}$$

B-4

The equation for  $\frac{\partial \beta}{\partial \underline{v}^{(B)}}$  is

$$\frac{\partial \beta}{\partial \underline{v}^{(B)}} = \begin{bmatrix} \frac{-v_{r_1} v_{r_2}}{v_m^3 \sqrt{v_{r_1}^2 + v_{r_3}^2}} \\ \frac{\sqrt{v_{r_1}^2 + v_{r_3}^2}}{v_m^3} \\ \frac{-v_{r_2} v_{r_3}}{v_m^3 \sqrt{v_{r_1}^2 + v_{r_3}^2}} \end{bmatrix}$$

B-5



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The following equations define the last three required partial derivatives

$$\frac{\partial \alpha}{\partial h} = - \frac{\partial \alpha}{\partial \underline{v}^{(B)}} B_C^{LL} \frac{\partial v_{-w}^{(LL)}}{\partial h} \quad B-6$$

$$\frac{\partial \beta}{\partial h} = - \frac{\partial \beta}{\partial \underline{v}^{(B)}} B_C^{LL} \frac{\partial v_{-w}^{(LL)}}{\partial h} \quad B-7$$

$$\frac{\partial v_m}{\partial h} = - \frac{\partial \beta}{\partial \underline{v}^{(B)}} B_C^{LL} \frac{\partial v_{-w}^{(LL)}}{\partial h} \quad B-8$$

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# APPENDIX C

PARTIAL DERIVATIVE OF THE VECTOR  ${}^B_C \mathbf{I} \underline{v}$  wrt  $\underline{\theta}$

The third of the frequently occurring required partial derivatives  
is

$$\frac{\partial}{\partial \theta} \begin{bmatrix} \cos \theta \cos \phi v_1 & + \cos \theta \sin \phi v_2 & - \sin \theta v_3 \\ (\sin \phi \sin \theta \cos \phi) v_1 & + (\sin \phi \sin \theta \sin \phi) v_2 & + \sin \phi \cos \theta v_3 \\ (\cos \phi \sin \theta \cos \phi) v_1 & + (\cos \phi \sin \theta \sin \phi) v_2 & + \cos \phi \cos \theta v_3 \end{bmatrix} \quad C-1$$

The resulting matrix is given in Table C-1.

TABLE C-1. Partial Derivative of  $B_{C^I}^I v$  wrt  $\theta$

$-\sin\theta\cos\psi v_1$	$-\cos\theta\sin\psi v_1$	0	
$-\sin\theta\sin\psi v_2$	$+\cos\theta\cos\psi v_2$		
$+\cos\theta v_3$			
$\sin\phi\cos\theta\cos\psi v_1$	$-(\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v_1$		$(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)v_1$
$+\sin\phi\cos\theta\sin\psi v_2$	$+(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v_2$		$+(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)v_2$
$-\sin\phi\sin\theta v_3$			$+\cos\phi\cos\theta v_3$
$\cos\phi\cos\theta\cos\psi v_1$	$-(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v_1$		$-(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v_1$
$+\cos\phi\cos\theta\sin\psi v_2$	$+(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)v_2$		$-(\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi)v_2$
$-\cos\phi\sin\theta v_3$			$-\sin\phi\cos\theta v_3$

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